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Lattice Uniformities on Effect Algebras

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Let L be a lattice ordered effect algebra. We prove that the lattice uniformities on L which make uniformly continuous the operations $\ominus$ and $\oplus$ of L are uniquely determined by their system of neighborhoods of 0 and form a distributive lattice. Moreover we prove that every such uniformity is generated by a family of weakly subadditive $[0, +\infty]$-valued functions on L.

KEY WORDS: effect algebras; lattice uniformities; submeasures.

1. INTRODUCTION

Effect algebras have been introduced by Foulis and Bennett (1994) for modeling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arise in quantum physics (see Beltrametti and Cassinelli, 1981) and in Mathematical Economics (see Butnariu and Klement, 1993; Epstein and Zhang, 2001), in particular of orthomodular lattices in non-commutative measure theory and MV-algebras in fuzzy measure theory. After 1994, there have been a great number of papers concerning effect algebras (see Dvurečenskij and Pulmannová, 2000, for a bibliography).

In this paper we study D-uniformities on a lattice ordered effect algebra L, i.e. lattice uniformities on L which make uniformly continuous the operations $\ominus$ and $\oplus$ of L.

The starting point of our paper is observing the key role played by D-uniformities in the study of modular measures on L (see Avallone, 2001; Avallone *et al.*, 2003; Avallone and Basile, 2003), since every modular measure on L generates a D-uniformity. Also of importance is the role played in the study of modular functions on orthomodular lattices (see Weber, 1995) and of measures on MV-algebras (see Barbieri and Weber, 1998; Graziano, 2000) by the lattice structure

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of filters which generate lattice uniformities making uniformly continuous the operations of these structures.

In the first part of the paper, we give a description of the filters which are systems of neighbourhoods of 0 in D-uniformities on L—called D-filters—and we prove that there exists an order isomorphism between the lattice of all D-uniformities on L and the lattice of all D-filters on L. In particular every D-uniformity is uniquely determined by its system of neighbourhoods of 0. As a consequence, we obtain that the lattice of all D-uniformities on L is distributive.

Our results extend similar results of Weber (1995) in orthomodular lattices (see also Avallone and Weber, 1997) and of Barbieri and Weber (1998) and Graziano (2000) in MV-algebras, and give as particular case the order isomorphism found in Avallone and Vitolo (2003) between some lattice congruences and some lattice ideals.

In the second part of the paper, we apply the results of the first part to prove that every D-uniformity on L is generated by a family of weakly subadditive $[0, +\infty]$-functions on L.

2. PRELIMINARIES

An *effect algebra* Dvurečenskij and Pulmannová, (2000) is a set E, with two distinguished elements 0 and 1, and a partially defined operation $\oplus$ such that for all $a, b, c \in E$:

(E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.
(E2) If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
(E3) There exists a unique $a^{\perp} \in E$ such that $a \oplus a^{\perp}$ is defined and $a \oplus a^{\perp} = 1$.
(E4) If $a \oplus 1$ is defined, then $a = 0$.

It is easily seen that $a \oplus 0$ is always defined and equals a. If $a \oplus b$ is defined, we say that a and b are *orthogonal* and write $a \perp b$.

In an effect algebra E another partially defined operation $\ominus$ can be defined by the following rule: $c \ominus a$ exists and equals b if and only if $a \oplus b$ exists and equals c. In particular, $a^{\perp} = 1 \ominus a$. Moreover, if $a \perp b$, then $a \oplus b = (a^{\perp} \ominus b)^{\perp} = (b^{\perp} \ominus a)^{\perp}$.

In an effect algebra E a partial ordering relation $\leq$ can be defined as follows: $a \leq c$ if and only if, for some $b \in E$, $a \oplus b$ exists and equals c. Hence $c \ominus a$ is defined if and only if $a \leq c$. Moreover $a \perp b$ if and only if $a \leq b^{\perp}$.

If $a \vee b$ and $a \wedge b$ exist for all $a, b \in E$, then we say that E is a *lattice ordered effect algebra* (otherwise called *D-lattice*). In this case, we define the *symmetric difference* of any two elements a and b in E as $a \triangle b = (a \vee b) \ominus (a \wedge b)$.

Throughout the paper, the symbol L will always denote a lattice ordered effect algebra. Let us recall that L is an *MV-algebra* if and only if $(a \vee b) \ominus b = a \ominus (a \wedge b)$ for all $a, b \in L$, while L is an *orthomodular lattice* if and only if $a^{\perp} \wedge a = 0$ for every $a \in L$.

We will make use of the following properties (for the proofs we refer to Dvurečenskij and Pulmannová, 2000).

Proposition 2.1. *For all $a, b, c \in L$ we have*

(i) *If $a \leq b$, then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$.*
(ii) *If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.*
(iii) *If $a \leq b \leq c$, then $b \ominus a \leq c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.*
(iv) *If $a \leq b^{\perp}$ and $a \oplus b \leq c$, then $c \ominus (a \oplus b) = (c \ominus a) \ominus b = (c \ominus b) \ominus a$.*
(v) *If $a \leq b \leq c^{\perp}$, then $a \oplus b \leq b \oplus c$ and $(b \oplus c) \ominus (a \oplus c) = b \ominus a$.*
(vi) *If $a \leq b \leq c$, then $a \oplus (c \ominus b) = c \ominus (b \ominus a)$.*
(vii) *If $a \leq b^{\perp} \leq c^{\perp}$, then $a \oplus (b \ominus c) = (a \oplus b) \ominus c$.*
(viii) *If $a \leq c$ and $b \leq c$, then $c \ominus (a \vee b) = (c \ominus a) \wedge (c \ominus b)$ and $c \ominus (a \wedge b) = (c \ominus a) \vee (c \ominus b)$.*
(ix) *If $c \leq a$ and $c \leq b$, then $(a \wedge b) \ominus c = (a \ominus c) \wedge (b \ominus c)$ and $(a \vee b) \ominus c = (a \ominus c) \vee (b \ominus c)$.*
(x) *If $a \leq c^{\perp}$ and $b \leq c^{\perp}$, then $(a \vee b) \oplus c = (a \oplus c) \vee (b \oplus c)$ and $(a \wedge b) \oplus c = (a \oplus c) \wedge (b \oplus c)$.*

Let $\mathcal{U}$ be a uniformity on L. We say that $\mathcal{U}$ is a *lattice uniformity* Weber (1991) if the operations $\vee$ and $\wedge$ are uniformly continuous with respect to $\mathcal{U}$.

A *D-uniformity* (Avallone, 2001) is a lattice uniformity which makes the operations $\oplus$ and $\ominus$ uniformly continuous, too. The set of all D-uniformities on L will be denoted by $\mathcal{DU}(L)$. It is easy to see that $\mathcal{DU}(L)$—ordered by inclusion—is a complete lattice, with the discrete uniformity and the trivial uniformity as greatest and smallest elements, respectively.

Given $U, V \subset L \times L$, we put

$$U \vee V = \{ (a_1 \vee b_1, a_2 \vee b_2) : (a_1, a_2) \in U,\ (b_1, b_2) \in V \},$$
$$U \wedge V = \{ (a_1 \wedge b_1, a_2 \wedge b_2) : (a_1, a_2) \in U,\ (b_1, b_2) \in V \},$$
$$U \ominus V = \{ (a_1 \ominus b_1, a_2 \ominus b_2) : b_1 \leq a_1,\ b_2 \leq a_2,\ (a_1, a_2) \in U,\ (b_1, b_2) \in V \}.$$

It is known (see Weber, 1991) that a uniformity $\mathcal{U}$ on L is a lattice uniformity if and only if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \vee \Delta \subset U$ and $V \wedge \Delta \subset U$, where $\Delta = \{ (a, a) : a \in L \}$.

Similarly, it has been shown in Avallone (2001) that a lattice uniformity $\mathcal{U}$ on L is a D-uniformity if and only if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \ominus \Delta \subset U$ and $\Delta \ominus V \subset U$.

3. *D*-UNIFORMITIES AND *D*-FILTERS

Definition 3.1. A filter $\mathcal{F}$ of subsets of a D-lattice L is called a *D-filter* if it satisfies the following:

(F1) $\forall F \in \mathcal{F} \quad \exists F' \in \mathcal{F}: \quad \forall a, b \in F' \quad [a \perp b \Longrightarrow a \oplus b \in F]$;
(F2) $\forall F \in \mathcal{F} \quad \exists G \in \mathcal{F}: \quad \forall a \in G \quad \forall c \in L \quad (a \vee c) \ominus c \in F$.

The set of all D-filters on L will be denoted by $\mathcal{FND}(L)$.

Note that, by 2.12.1, a filter $\mathcal{F}$ satisfies 3.1. if and only if, for every $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ such that, for all $a \in G$ and all $c \in L$, one has $c \ominus (a^{\perp} \wedge c) \in F$.

We shall prove, in Theorem 3.4 below, that $\mathcal{FND}(L)$ is isomorphic to $\mathcal{DU}(L)$ and that $\mathcal{F}$ is a D-filter if and only if $\mathcal{F}$ is the system of neighbourhoods of 0 in a D-uniformity.

Lemma 3.2. *For every $a, b, c, d \in L$ such that $c \leq a$, $c \leq b$, $d \geq a$ and $d \geq b$ one has $(a \ominus c)\Delta(b \ominus c) = a\Delta b = (d \ominus a)\Delta(d \ominus b)$.*

Proof: Indeed, applying 2.12.1, and 2.12.1, one gets $(a \ominus c)\Delta(b \ominus c) = ((a \ominus c) \vee (b \ominus c)) \ominus ((a \ominus c) \wedge (b \ominus c)) = ((a \vee b) \ominus c) \ominus ((a \wedge b) \ominus c) = (a \vee b) \ominus (a \wedge b) = a\Delta b$. Similarly, applying 2.12.1, and 2.12.1, one gets $(d \ominus a)\Delta (d \ominus b) = ((d \ominus a) \vee (d \ominus b)) \ominus ((d \ominus a) \wedge (d \ominus b)) = (d \ominus (a \wedge b)) \ominus (d \ominus (a \vee b)) = (a \vee b) \ominus (a \wedge b) = a\Delta b$. □

Proposition 3.3. *A D-filter $\mathcal{F}$ on L has the following properties:*

(i) $\forall F \in \mathcal{F} \quad \exists G \in \mathcal{F}: \quad \forall a \in G \quad \forall b \in L \quad [b \leq a \Longrightarrow b \in F]$;
(ii) $\forall F \in \mathcal{F} \quad \exists G \in \mathcal{F}: \quad \forall a, b \in G \quad a \vee b \in F$;
(iii) $\forall F \in \mathcal{F} \quad \exists G \in \mathcal{F}: \quad \forall x, y, z \in L \quad [x\Delta y \in G \Longrightarrow (x \vee z)\Delta(y \vee z) \in F]$;
(iv) $\forall F \in \mathcal{F} \quad \exists G \in \mathcal{F}: \quad \forall x, y, z \in L \quad [x\Delta y \in G \Longrightarrow (x \wedge z)\Delta(y \wedge z) \in F]$;
(v) $\forall F \in \mathcal{F} \quad \exists G \in \mathcal{F}: \quad \forall x, y, z \in L \ [x\Delta y \in G, \quad y\Delta z \in G \Longrightarrow x\Delta z \in F]$.

Proof:

(i) Let $F \in \mathcal{F}$ and let $G \in \mathcal{F}$ such that 3.1. is satisfied. Given any $a \in G$ and any $b \in L$ with $b \leq a$, put $c = a \ominus b$. Then $b = a \ominus (a \ominus b) = (a \vee (a \ominus b)) \ominus (a \ominus b) = (a \vee c) \ominus c \in F$.

(ii) Given $F \in \mathcal{F}$, let $F' \in \mathcal{F}$ satisfy 3.1., and let $G \in \mathcal{F}$ satisfy 3.1. with F' in place of F. If $a, b \in G$, then $(a \vee b) \ominus b \in F'$. Moreover $b \in F'$ by 3.3. Therefore $a \vee b = ((a \vee b) \ominus b) \oplus b \in F$.

(iii) Let $F \in \mathcal{F}$ and let $G \in \mathcal{F}$ such that 3.1. is satisfied. Given x, y, z such that $x \triangle y \in G$, we put $a = x \triangle y$ and $c = ((x \vee z) \wedge (y \vee z)) \ominus (x \wedge y)$ and we show that $(x \vee z) \triangle (y \vee z) = (a \vee c) \ominus c$. First observe that $x \vee y \vee z = (x \vee y) \vee ((x \vee z) \wedge (y \vee z))$. Now, applying 2.12.1 and 2.12.1, we have:

$$\begin{aligned}(x \vee z) \triangle (y \vee z) &= (x \vee y \vee z) \ominus ((x \vee z) \wedge (y \vee z))\\ &= ((x \vee y) \vee ((x \vee z) \wedge (y \vee z))) \ominus ((x \vee z) \wedge (y \vee z))\\ &= (((x \triangle y) \oplus (x \wedge y)) \vee ((x \vee z) \wedge (y \vee z))) \ominus ((x \vee z) \wedge (y \vee z))\\ &= ((a \oplus (x \wedge y)) \vee (c \oplus (x \wedge y))) \ominus (c \oplus (x \wedge y))\\ &= ((a \vee c) \oplus (x \wedge y)) \ominus (c \oplus (x \wedge y)) = (a \vee c) \ominus c.\end{aligned}$$

(iv) Given $F \in \mathcal{F}$, take $G \in \mathcal{F}$ such that 3.3 is satisfied, and let x, y, z such that $x \triangle y \in G$. By Lemma 3.2 we have $x^{\perp} \triangle y^{\perp} = x \triangle y$, and therefore $(x \wedge z) \triangle (y \wedge z) = (x^{\perp} \vee z^{\perp})^{\perp} \triangle (y^{\perp} \vee z^{\perp})^{\perp} = (x^{\perp} \vee z^{\perp}) \triangle (y^{\perp} \vee z^{\perp}) \in F$.

(v) Given $F \in \mathcal{F}$, let $F_1 \in \mathcal{F}$ satisfy 3.3, let $F_2 \in \mathcal{F}$ satisfy 3.3 with F_1 in place of F, let $F_3 \in \mathcal{F}$ satisfy 3.3 with F_2 in place of F and let $G \in \mathcal{F}$ satisfy 3.3 with F_3 in place of F. If $a, b, c \in L$ are such that both $x \triangle y$ and $x \triangle z$ belong to G, then $a = (x \vee y \vee z) \ominus (y \vee z) = ((x \vee (x \vee z)) \triangle (y \vee (y \vee z)) \in F_3$ and $b = (y \vee z) \ominus z = (y \vee z) \triangle (z \vee z) \in F_3$ also. It follows that $(x \vee y \vee z) \ominus z = a \oplus b \in F_2$, so that $(x \vee z) \ominus z \in F_1$. Similarly one shows that $(x \vee z) \ominus x \in F_1$. Hence $x \triangle z = ((x \vee z) \ominus z) \vee ((x \vee z) \ominus x) \in F$. □

Theorem 3.4.

(a) If $\mathcal{U}$ is a D-uniformity, then the filter $\mathcal{F}_{\mathcal{U}}$ of neighbourhoods of 0 *in $\mathcal{U}$ is a D-filter.*

(b) Let $\mathcal{F}$ be a D-filter and, for each $F \in \mathcal{F}$, let $F^{\triangle} = \{(a, b) \in L \times L : a \triangle b \in F\}$. Then $\mathcal{B} = \{F^{\triangle} : F \in \mathcal{F}\}$ is a base for a D-uniformity whose filter of neighbourhoods of 0 *is $\mathcal{F}$.*

(c) The mapping $\Psi: \mathcal{U} \mapsto \mathcal{F}_{\mathcal{U}}$ is an order-isomorphism of $\mathcal{DU}(L)$ onto $\mathcal{FND}(L)$ (both ordered by inclusion).

Proof:

(a) Since $\oplus$ is continuous at (0, 0), for every $F \in \mathcal{F}_{\mathcal{U}}$ there exists $F' \in \mathcal{F}_{\mathcal{U}}$ such that if $(a, b) \in F' \times F'$ and $a \perp b$, then $a \oplus b \in F$. This gives 3.1. To prove 3.1., let $F \in \mathcal{F}_{\mathcal{U}}$ and let $U \in \mathcal{U}$ with $U(0) \subseteq F$. By uniform continuity of $\ominus$ and $\vee$, there exist $V_1, V_2 \in \mathcal{U}$ such that $V_1 \ominus \Delta \subset U$ and $V_2 \vee \Delta \subset V_1$. Now put $G = V_2(0)$, and consider any $a \in G$. Then $(0, a) \in V_2$, so that for every $c \in L$ we have $(c, a \vee c) \in V_2 \vee \Delta \subset V_1$ and hence $(0, (a \vee c) \ominus c) \in V_1 \ominus \Delta \subset U$ which means that $(a \vee c) \ominus c \in U(0) \subseteq F$.

(b) Clearly $F^{\triangle}$ is symmetric and $\Delta \subset F^{\triangle}$ for every $F \in \mathcal{F}$. Moreover, given $F_1, F_2 \in \mathcal{F}$, let $F_3 = \mathcal{F}_1 \cap F_2$. Then $F_3^{\triangle} = F_1^{\triangle} \cap F_2^{\triangle}$. Finally, if $F \in \mathcal{F}$ and $G \in \mathcal{F}$ satisfies 3.33.3, we have that $G^{\triangle} \circ G^{\triangle} \subseteq F^{\triangle}$. Therefore $\mathcal{B}$ is a base for a uniformity $\mathcal{U}$.

Now fix $U \in \mathcal{U}$. We show that there exists $V \in \mathcal{U}$ such that both $V \vee \Delta$ and $V \wedge \Delta$ are contained in U. Let $G \in \mathcal{F}$ satisfy 3.33.3 and put $V = G^{\triangle}$. Given $(x, y) \in V \vee \Delta$, take $a, b, c \in L$ with $x = a \vee c$, $y = b \vee c$ and $(a, b) \in V$, that is $a \triangle b \in G$. By 3.33.3, we have $x \triangle y = (a \vee c) \triangle (b \vee c) \in F$, that is $(x, y) \in F^{\triangle}$. We conclude that $V_1 \vee \Delta \subset F^{\triangle} \subseteq U$. Since the same G also satisfies 3.33.3 one sees in a similar way that $V \wedge \Delta \subset F^{\triangle} \subseteq U$ too. Next, we show that there exists $V \in \mathcal{U}$ such that both $V \ominus \Delta$ and $\Delta \ominus V$ are contained in U. Choose $F \in \mathcal{F}$ such that $F^{\triangle} \subseteq U$ and put $V = F^{\triangle}$. By Lemma 3.2, one has $F^{\triangle} \ominus \Delta = \{ (a \ominus c, b \ominus c) : c \leq a,\ c \leq b,\ a \triangle b \in F \} = \{ (a \ominus c, b \ominus c) : c \leq a,\ c \leq b,\ (a \ominus c) \triangle (b \ominus c) \in F \} = F^{\triangle}$ and similarly one sees that $\Delta \ominus F^{\triangle} = F^{\triangle}$. Hence $V \ominus \Delta \subset U$ and $\Delta \ominus V \subset U$.

It remains to prove that the filter of neighbourhoods of 0 in $\mathcal{U}$ coincides with $\mathcal{F}$. First observe that, given any $F \in \mathcal{F}$, we have

$$F^{\triangle}(0) = \{a \in L : (0, a) \in F^{\triangle}\} = \{a \in L : a \triangle 0 \in F\} = F \qquad (1)$$

and therefore F is a neighbourhood of 0 in $\mathcal{U}$. Conversely, if G is a neighbourhood of 0 in $\mathcal{U}$, since $\mathcal{B}$ is a base for $\mathcal{U}$, there exists $F \in \mathcal{F}$ such that $F^{\triangle}(0) \subseteq G$. By 1, this means that $F \subset G$ and hence $G \in \mathcal{F}$, because $\mathcal{F}$ is a filter.

(c) It follows from 3.4 that Ψ maps $\mathcal{DU}(L)$ into $\mathcal{FND}(L)$. Now for any $\mathcal{F} \in \mathcal{FND}(L)$ let $\Phi(\mathcal{F})$ denote the D-uniformity constructed as in 3.4. Since $\Psi(\Phi(\mathcal{F})) = \mathcal{F}$, we have that Ψ is onto. Moreover if $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{DU}(L)$ and $\mathcal{F}_1 \subset \mathcal{F}_2$, then $\{F^{\triangle} : F \in \mathcal{F}_1\} \subseteq \{F^{\triangle} : F \in \mathcal{F}_1\}$ whence $\Phi(\mathcal{F}_1) \subseteq \Phi(\mathcal{F}_2)$. On the other hand, if $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{DU}(L)$ and $\mathcal{U}_1 \subset \mathcal{U}_2$, then the topology induced by $\mathcal{U}_1$ is coarser than the one induced by $\mathcal{U}_2$, hence $\Psi(\mathcal{U}_1) \subseteq \Psi(\mathcal{U}_2)$.

Finally we show that $\Phi = \Psi^{-1}$, so that Ψ is one-to-one. Given $\mathcal{F} \in \mathcal{FND}(L)$, we consider any $\mathcal{U} \in \mathcal{DU}(L)$ such that $\mathcal{F} = \Psi(\mathcal{U})$ and prove that $\Phi(\mathcal{F}) = \mathcal{U}$. If $F \in \mathcal{F}$, then it is a neighbourhood of 0, hence there is $U \in \mathcal{U}$ such that $U(0) \subseteq F$. By uniform continuity of $\triangle$, there exists $V \in \mathcal{U}$ with $V \triangle \Delta \subset U$. Now let $(a, b) \in V$. We have $(0, a \triangle b) = (a \triangle a, b \triangle a) \in V \triangle \Delta \subset U$, whence $a \triangle b \in U(0) \subseteq F$.

Hence $V \subset F^{\triangle}$ and therefore $\mathcal{U}$ is finer than $\Phi(\mathcal{F})$. Conversely let $U \in \mathcal{U}$. Consider a symmetric $V_1 \in \mathcal{U}$ with $V_1 \circ V_1 \subset U$, and take $V_2, V_3 \in \mathcal{U}$ such that $V_2 \vee \Delta \subset V_1$ and $V_3 \oplus \Delta \subset V_2$. Put $F = V_3(0)$, so that $F \in \mathcal{F}$. If $(a, b) \in F^{\triangle}$, we have $a\triangle b \in F$, that is $(0, a\triangle b) \in V_3$. It follows that $(a \wedge b, a \vee b) = (0 \oplus (a \wedge b), (a\triangle b) \oplus (a \wedge b)) \in V_3 \oplus \Delta \subset V_2$, hence $(a, a \vee b) = ((a \wedge b) \vee a, (a \vee b) \vee a) \in V_2 \vee \Delta \subset V_1$ and, similarly $(b, a \vee b) \in V_1$. Since $V_1^{-1} = V_1$ we also have $(a \vee b, b) \in V$, and then $(a, b) \in V_1 \circ V_1 \subset U$. Therefore $F^{\triangle} \subseteq U$. We conclude that $\mathcal{U} \subset \Phi(\mathcal{F})$, whence the equality. □

The reader should note that the above theorem implies, as particular cases, the results of Barbieri and Weber (1998, Theorem 2.1) and Graziano (2000, Theorem 3.6) for MV-algebras, as well as Weber (1995, Theorem 1.1) for orthomodular lattices.

From Theorem 3.43.4, by restricting to principal filters, one can deduce the order isomorphism between D-congruences and D-ideals, which has been found using a different approach (Avallone and Vitolo, in 2003, Theorem 4.5).

Proposition 3.5. *Let $\mathcal{F}$ be the filter of neighbourhoods of 0 in a D-uniformity $\mathcal{U}$. For every $F \in \mathcal{F}$, let $F^{\oplus} = \{ (a, b) \in L \times L : \exists h, k \in F : h \perp a,\ k \perp b,\ a \oplus h = b \ominus k \}$ and $F^{\ominus} = \{ (a, b) \in L \times L : \exists i, j \in F : i \leq a,\ j \leq b,\ a \ominus i = b \ominus j \}$. Then both $\{ F^{\oplus} : F \in \mathcal{F} \}$ and $\{ F^{\ominus} : F \in \mathcal{F} \}$ are bases for $\mathcal{U}$.*

Proof: It suffices to show that, for every $F \in \mathcal{F}$, there exist $F_1, F_2 \in \mathcal{F}$ such that $F^{\oplus}, F^{\ominus} \supseteq F_1^{\triangle}$ and $F_2^{\oplus}, F_2^{\ominus} \subseteq F^{\triangle}$.

Let $F_1 \in \mathcal{F}$ satisfy 3.33.3. Given $(a, b) \in F_1^{\triangle}$, we put $h = (a \vee b) \ominus a$, $k = (a \vee b) \ominus b$, $i = a \ominus (a \wedge b)$ and $j = b \ominus (a \wedge b)$. Since $h \leq (a \vee b) \ominus (a \wedge b) = a\triangle b \in F_1$, we have $h \in F$. In the same way one sees that k, i and j belong to F, too. Moreover we have $a \oplus h = a \oplus ((a \vee b) \ominus a) = a \vee b = b \oplus ((a \vee b) \ominus b) = b \oplus k$, so that $(a, b) \in F^{\oplus}$. Similarly, applying 2.12.1, we have $a \ominus i = a \ominus (a \ominus (a \wedge b)) = a \wedge b = b \ominus (b \ominus (a \wedge b)) = b \ominus j$, so that $(a, b) \in F^{\ominus}$.

Now let $G \in \mathcal{F}$ satisfy 3.33.3, and take $F_2 \in \mathcal{F}$ satisfying 3.33.3 with G in place of F. Given $(a, b) \in F_2^{\oplus}$, there are $h, k \in F_2$ such that $h \perp a$, $k \perp b$ and $a \oplus h = b \oplus k$. Since $a \vee b \leq (a \oplus h) \vee (b \oplus k) = a \oplus h = b \oplus k$, we get $(a \vee b) \ominus a \leq h$ and $(a \vee b) \ominus b \leq k$, so that both $(a \vee b) \ominus a$ and $(a \vee b) \ominus b$ belong to G. By 2.12.1, we have $a\triangle b = ((a \vee b) \ominus a) \vee ((a \vee b) \ominus b)$ hence $a\triangle b \in F$, i.e. $(a, b) \in F^{\triangle}$. Similarly, given $(a, b) \in F_2^{\ominus}$, take $i, j \in F_2$ such that $i \leq a$, $j \leq b$ and $a \ominus i = b \ominus j$. Observe that $a \ominus i = (a \ominus i) \wedge (b \ominus j) \leq a \wedge b$ thus, applying 2.12.1, $i = a \ominus (a \ominus i) \geq a \ominus (a \wedge b)$. It follows that $a \ominus (a \wedge b) \in G$, and in the same way one sees that $b \ominus (a \wedge b) \in G$, too. By 2.12.1, we have $a\triangle b = (a \ominus (a \wedge b)) \vee (b \ominus (a \wedge b))$ hence $a\triangle b \in F$, i.e. $(a, b) \in F^{\triangle}$. □

Given $F, G \subset L$, we will put $F \oplus G = \{ f \oplus g : f \perp g,\ f \in F,\ g \in G \}$. Using this notation, condition 3.1 may be rewritten as follows: $\forall F \in \mathcal{F} \quad \exists F' \in \mathcal{F} : F' \oplus F' \subseteq F$.

Proposition 3.6.

(a) If $\mathcal{F}, \mathcal{G} \in \mathcal{FND}(L)$, then $\{ F \oplus G : F \in \mathcal{F},\ G \in \mathcal{G} \}$ is a base for $\mathcal{F} \wedge \mathcal{G}$ in $\mathcal{FND}(L)$.

(b) If $\Gamma \subset \mathcal{FND}(L)$, then $\bigvee \Gamma$ in $\mathcal{FND}(L)$ is the set of all intersections of finite subsets of $\bigcup \Gamma$. In particular $\mathcal{G}_1 \vee \mathcal{G}_2 = \{ G_1 \cap G_2 : G_1 \in \mathcal{G}_1,\ G_2 \in \mathcal{G}_2 \}$ for all $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{FND}(L)$.

Proof:

(a) First observe that

$$\forall F \in \mathcal{F} \quad \forall G \in \mathcal{G} \quad F \cup G \subset F \oplus G. \tag{2}$$

Indeed, since $0 \in G$, one has $F = \{ f \oplus 0 : f \in F \} \subseteq \{ f \oplus g : f \perp g,\ f \in F,\ g \in G \} = F \oplus G$, and similarly for G. In particular, all sets $F \oplus G$ with $F \in \mathcal{F}$ and $G \in \mathcal{G}$ are nonempty. Now, given $F_1 \oplus G_1$ and $F_2 \oplus G_2$, with $F_1, F_2 \in \mathcal{F}$ and $G_1, G_2 \in \mathcal{G}$, let $F = F_1 \cap F_2$ and $G = G_1 \cap G_2$. We have $F \oplus G = \{ f \oplus g : f \perp g,\ f \in F,\ g \in G \} \subseteq \{ f \oplus g : f \perp g,\ f \in F_1,\ g \in G_1 \} = F_1 \oplus G_1$ and, similarly, $F \oplus G \subset F_2 \oplus G_2$. Hence $F \oplus G \subset (F_1 \oplus G_1) \cap (F_2 \oplus G_2)$. Therefore $\{ F \oplus G : F \in \mathcal{F},\ G \in \mathcal{G} \}$ is a base for a filter which we denote by $\mathcal{H}$.

We prove that $\mathcal{H}$ is a D-filter. Given any $H \in \mathcal{H}$, let $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \oplus G \subset H$. Take $F', F'' \in \mathcal{F}$ satisfying 3.1. and 3.1. respectively, and choose $G', G'' \in \mathcal{G}$ in a similar way. Clearly $H' = F' \oplus G'$ and $H'' = F'' \oplus G''$ belong to $\mathcal{H}$. We show that H' satisfies 3.1. and H'' satisfies 3.1. (with H in place of F). If a and b are orthogonal elements of H', then $a = f_1 \oplus g_1$ and $b = f_2 \oplus g_2$, where $f_1, f_2 \in F'$ and $g_1, g_2 \in G'$. Note that $f_1 \perp f_2$ and $g_1 \perp g_2$, hence $f = f_1 \oplus f_2 \in F$ and $g = g_1 \oplus g_2 \in G$. Therefore $a \oplus b = (f_1 \oplus g_1) \oplus (f_2 \oplus g_2) = (f_1 \oplus f_2) \oplus (g_1 \oplus g_2) = f \oplus g \in F \oplus G \subset H$. Now let $a \in H''$ and $c \in L$. Let $f \in F''$ and $g \in G''$ such that $a = f \oplus g$, and put $d = (f \vee c) \ominus f$. We have $f' = (f \vee c) \ominus c \in F$ and $g' = (g \vee d) \ominus d \in G$. Since $g \vee d = g' \oplus d$ and $f \vee c = f \oplus d$, applying 2.12.1 and 2.12.1, we obtain $(a \vee c) \ominus c = (a \vee f \vee c) \ominus c = ((f \oplus g) \vee (f \vee c)) \ominus c = ((f \oplus g) \vee (f \oplus d)) \ominus c = (f \oplus (g \vee d)) \ominus c = (f \oplus (g \vee d)) \ominus c = (f \oplus (g' \oplus d)) \ominus c = ((f \oplus d) \oplus g') \ominus c = ((f \vee c) \oplus g') \ominus c = ((f \vee c) \ominus c) \oplus g' = f' \oplus g' \in F \oplus G \subset H$.

It follows from 2 that both $\mathcal{F}$ and $\mathcal{G}$ are finer than $\mathcal{H}$. To complete the proof, consider any D-filter $\mathcal{H}'$ such that both $\mathcal{F}$ and $\mathcal{G}$ are finer than $\mathcal{H}'$. We show that $\mathcal{H}' \subseteq \mathcal{H}$. Let $H \in \mathcal{H}$. By 3.1, there exists $H' \in \mathcal{H}'$ such that $H' \oplus H' \subseteq H$. Since $H' \in \mathcal{F} \cap \mathcal{G}$ we get $H' \oplus H' \in \mathcal{H}$ and hence $H \in \mathcal{H}$, too.

(b) Let $\mathcal{F}$ be the set of intersections of finite subsets of $\bigcup \Gamma$. We show that $\mathcal{F}$ is a filter.

Let $F_1, F_2 \in \mathcal{F}$. One has $F_1 = \bigcap \mathcal{F}_1$ and $F_1 = \bigcap \mathcal{F}_1$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are finite subsets of $\bigcup \Gamma$. If $G = F_1 \cap F_2$, then $G \in \mathcal{F}$ because it is the intersection of $\mathcal{F}_1 \cup \mathcal{F}_2$, which is again a finite subset of $\bigcup \Gamma$. Now let $F \in \mathcal{F}$. Then $F = \bigcap_{i=1}^n F_i$, where $F_i \in \mathcal{G}_i$ and $\mathcal{G}_i \in \Gamma$ for each $i \in \{1, 2, \dots, n\}$. If $G \supset F$, let $A = G \setminus F$. For each i, one has $G_i = A \cup F_i \in \mathcal{G}_i$, and $\bigcap_{i=1}^n G_i = \bigcap_{i=1}^n (A \cup F_i) = A \cup \bigcap_{i=1}^n F_i = A \cup F = G$. Hence $G \in \mathcal{F}$.

Now we check properties 3.1 and 3.1. Let $F \in \mathcal{F}$. As above, $F = \bigcap_{i=1}^n F_i$, with $F_i \in \mathcal{G}_i \in \Gamma$. For each i, take F_i' and G_i in $\mathcal{G}_i$ satisfying 3.1. and 3.1. respectively (with F_i in place of F). Put $F' = \bigcap_{i=1}^n F_i'$ and $G = \bigcap_{i=1}^n G_i$. Clearly F' and G belong to $\mathcal{F}$. We show that F' satisfies 3.1. and G satisfies 3.1. If a and b are orthogonal elements of F', then for each $i \in \{1, 2, \dots, n\}$ we have $a, b \in F_i'$ and hence $a \oplus b \in F_i$. Therefore $a \oplus b \in F$. Similarly, if $a \in G$ and $c \in L$, then for each $i \in \{1, 2, \dots, n\}$ we have $a \in G_i$ and hence $(a \vee c) \ominus c \in F_i$. Therefore $(a \vee c) \ominus c \in F$.

Since it is clear that each $\mathcal{G} \in \Gamma$ is contained in $\mathcal{F}$ (indeed every G in $\mathcal{G}$ is the intersection of $\{G\}$, which a finite subset of $\bigcup \Gamma$), it remains to prove that any D-filter which is finer than all filters in Γ is finer than $\mathcal{F}$, too. So let $\mathcal{G}' \in \mathcal{FND}(L)$ such that $\mathcal{G} \subset \mathcal{G}'$ for every $\mathcal{G} \in \Gamma$. Given $F \in \mathcal{F}$, one has $F = \bigcap_{i=1}^n F_i$ where $F_i \in \mathcal{G}_i \in \Gamma$, hence $F_i \in \mathcal{G}'$, for each $i \in \{1, 2, \dots, n\}$. Since $\mathcal{G}'$ is a filter, we have $F \in \mathcal{G}'$. We conclude that $\mathcal{F} \subset \mathcal{G}'$ □

Corollary 3.7. *$\mathcal{DU}(L)$ and $\mathcal{FND}(L)$ are distributive (complete) lattices.*

Proof: By Theorem 3.43.4, it is enough to consider $\mathcal{FND}(L)$. Let $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}$ be D-filters. We have to verify that $(\mathcal{F} \vee \mathcal{G}_1) \wedge (\mathcal{F} \vee \mathcal{G}_2) \subseteq \mathcal{F} \vee (\mathcal{G}_1 \wedge \mathcal{G}_2)$.

Given $H \in (\mathcal{F} \vee \mathcal{G}_1) \wedge (\mathcal{F} \vee \mathcal{G}_2)$, take $F_1, F_2 \in \mathcal{F}$ and $G_1, G_2 \in \mathcal{G}$ with $(F_1 \cap G_1) \oplus (F_2 \cap G_2) \subseteq H$. Put $F = F_1 \cap F_2$ and let $F' \in \mathcal{F}$ satisfying 3.33.3. We complete the proof by showing that $F' \cap (G_1 \oplus G_2) \subseteq (F_1 \cap G_1) \oplus (F_2 \cap G_2)$.

Let $a \in F' \cap (G_1 \oplus G_2)$. Choose $a_1 \in G_1$ and $a_2 \in G_2$ such that $a = a_1 \oplus a_2$. Since $a_1 \leq a$ and $a \in F'$, one has $a_1 \in F \subset F_1$ and hence $a_1 \in F_1 \cap G_1$.

Similarly one sees that $a_2 \in F_2 \cap G_2$. Therefore $a = a_1 \oplus a_2 \in (F_1 \cap G_1) \oplus (F_2 \cap G_2)$. □

Proposition 3.8. *If $\mathcal{F}, \mathcal{G} \in \mathcal{FND}(L)$, then $\{ F \wedge G : F \in \mathcal{F},\ G \in \mathcal{G} \}$ is a base for $\mathcal{F} \vee \mathcal{G}$, where $F \wedge G = \{ f \wedge g : f \in F,\ g \in G \}$.*

Proof: Given $F \in \mathcal{F}$ and $G \in \mathcal{G}$, since $F \cap G = \{ a \wedge a : a \in F \cap G \} \subseteq \{ f \wedge g : f \in F,\ g \in G \} = F \wedge G$, it remains to prove that there exist $F' \in \mathcal{F}$ and $G' \in \mathcal{G}$ such that $F' \wedge G' \subseteq F \cap G$. Take $F' \in \mathcal{F}$ satisfying 3.33.3, and let G' be a member of $\mathcal{G}$ satisfying 3.33.3 also, but with G in place of F. If $f \in F'$ and $g \in G'$, then $f \wedge g \leq f$ hence $f \wedge g \in F$ and, similarly, $f \wedge g \leq g$ hence $f \wedge g \in G$. Therefore $f \wedge g \in F \cap G$. □

4. GENERATING *D*-UNIFORMITIES BY MEANS OF *K*-SUBMEASURES

Definition 4.1. Let $k \geq 1$. We say that a function $\eta\colon L \to [0, +\infty]$ is a *k-submeasure* if the following conditions hold

(S1) $\eta(0) = 0$;
(S2) $\forall a, b \in L \quad [a \leq b \Longrightarrow \eta(a) \leq \eta(b)]$;
(S3) $\forall a, b \in L \quad [a \perp b \Longrightarrow \eta(a \oplus b) \leq k\eta(a) + \eta(b)]$;
(S4) $\forall a, b \in L \quad \eta((a \vee b) \ominus b) \leq k\eta(a)$.

A 1-submeasure is simply called a *submeasure*.

Observe that, if L is an MV-algebra, then every function $\eta\colon L \to [0, +\infty]$ satisfying 4.1, 4.1, and 4.1 with $k = 1$ is a submeasure.

For every $\varepsilon > 0$, put $S_\varepsilon = \{ (x, y) \in [0, +\infty[\times [0, +\infty[: |x - y| < \varepsilon \} \cup \{(+\infty, +\infty)\}$. Then $\{ S_\varepsilon : \varepsilon > 0 \}$ is base for a uniformity $\mathcal{S}$ on $[0, +\infty]$ whose relativization to $[0, +\infty[$ is the usual uniformity, while $+\infty$ is a uniformly isolated point. In the sequel we will endow $[0, +\infty]$ with this uniformity.

Proposition 4.2. *For every k-submeasure η there exists a D-uniformity $\mathcal{U}(\eta)$ which is the weakest D-uniformity making η uniformly continuous.*

Proof: For each $\varepsilon > 0$, let $F_\varepsilon = \{ a \in L : \eta(a) < \varepsilon \}$. Since $F_{\varepsilon_1} \cap F_{\varepsilon_2} = F_{\min\{\varepsilon_1, \varepsilon_2\}}$, the collection $\{ F_\varepsilon : \varepsilon > 0 \}$ is a base for a filter $\mathcal{F}$. We show that $\mathcal{F}$ is a D-filter. Fix F in $\mathcal{F}$, and take $\varepsilon > 0$ with $F_\varepsilon \subset F$. Then $F' = F_{\frac{\varepsilon}{k+1}}$ satisifies 3.1 and $G = F_{\frac{\varepsilon}{k}}$ satisfies 3.1.

From Theorem 3.43.4, the sets $F_\varepsilon^\triangle$ form a base for a D-uniformity $\mathcal{U}(\eta)$. Now we show that η is $\mathcal{U}(\eta)$-uniformly continuous. Let $\varepsilon > 0$ and choose $\delta = \frac{\varepsilon}{k}$. For every $(a, b) \in F_\delta^\triangle$, we have $\eta(a \vee b) = \eta((a \triangle b) \oplus (a \wedge b)) \leq k\eta(a \triangle b) + \eta(a \wedge b) < \eta(a \wedge b) + k\delta = \eta(a \wedge b) + \varepsilon$. Thus, if $\eta(a \vee b) = +\infty$, then $\eta(a \wedge b) = +\infty$ whence, by monotonicity, $\eta(a) = \eta(b) = +\infty$. Otherwise, again by monotonicity, $\eta(a)$ and $\eta(b)$ are both finite, and moreover $|\eta(a) - \eta(b)| \leq |\eta(a \vee b) - \eta(a \wedge b)| < \varepsilon$. Hence, in any case, $(\eta(a), \eta(b)) \in S_\varepsilon$.

Finally, let $\mathcal{V}$ be a D-uniformity on L making η uniformly continuous. We prove that $\mathcal{U}(\eta) \leq \mathcal{V}$, which, by Theorem 3.43.4, is equivalent to $\mathcal{F} \subset \mathcal{G}$, where $\mathcal{G}$ is the filter of neighbourhoods of 0 in $\mathcal{V}$. Take any $F \in \mathcal{F}$, and choose $\varepsilon > 0$ with $F_\varepsilon \subset F$. Since η is continuous at 0 with respect to $\mathcal{V}$, and $\eta(0) = 0$, there is some $G \in \mathcal{G}$ such that if $a \in G$ then $\eta(a) < \varepsilon$, i.e. $a \in F_\varepsilon$. It follows that $G \subset F_\varepsilon \subset F$, hence $F \in G$. □

Our aim is to prove a sort of converse of the previous result, namely Theorem 4.4 below.

Proposition 4.3. *Let $k, m \geq 1$, and d be a pseudometric such that for all a, b, $c \in L$:*

(P1) $d(a \wedge c, b \wedge c) \leq d(a, b)$;
(P2) $a \perp c,\ b \perp c \Longrightarrow d(a \oplus c, b \oplus c) \leq kd(a, b)$;
(P3) $d((a \vee c) \ominus c, (b \vee c) \ominus c) \leq md(a, b)$;
(P4) $d((a \vee c) \ominus c, 0) \leq kd(a, 0)$.

For each $a \in L$, put $\tilde{\eta}(a) = d(a, 0)$. Then $\tilde{\eta}$ is a k-submeasure and $\mathcal{U}(\tilde{\eta})$ coincides with the uniformity induced by d.

Proof: It is clear that $\tilde{\eta}$ satisfies 4.1. Moreover, if $a \leq b$, by 4.3 we have $\tilde{\eta}(a) = d(a, 0) = d(b \wedge a, 0 \wedge a) \leq d(b, 0) = \tilde{\eta}(b)$ and 4.1 is proved. Now if $a, b \in L$ are orthogonal, then, applying the triangular inequality and 4.3, we get $\tilde{\eta}(a \oplus b) = d(a \oplus b, 0) \leq d(a \oplus b, b) + d(b, 0) \leq kd(a, 0) + d(b, 0) = k\tilde{\eta}(a) + \tilde{\eta}(b)$, that is 4.1. Similarly, taking any $a, b \in L$, by 4.3 we get $\tilde{\eta}((a \vee b) \ominus b) = d((a \vee b) \ominus b, 0) \leq kd(a, 0) = k\tilde{\eta}(a)$, that is 4.1.

Denote by $\mathcal{V}$ the uniformity induced by d. The sets $V_\varepsilon = \{ (a, b) \in L \times L : d(a, b) < \varepsilon \}$ form a base for $\mathcal{V}$, while the sets $F_\varepsilon^\triangle = \{ (a, b) \in L \times L : \tilde{\eta}(a \triangle b) < \varepsilon \}$ form a base for $\mathcal{U}(\tilde{\eta})$, as we have seen in Proposition 4.2. We show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $F_\delta^\triangle \subseteq V_\varepsilon$ and $V_\delta \subset F_\varepsilon^\triangle$. This will prove that $\mathcal{V} = \mathcal{U}(\tilde{\eta})$.

Take $\delta = \frac{\varepsilon}{2km}$. Given $(a, b) \in F_\delta^\triangle$, applying 4.3 and 4.3, we have $d(a, b) \leq d(a, a \wedge b) + d(a \wedge b, b) = d((a \vee b) \wedge a, (a \wedge b) \wedge a) + d((a \wedge b) \wedge b, (a \vee b) \wedge b) \leq 2d(a \vee b, a \wedge b) = 2d((a \triangle b) \oplus (a \wedge b), 0 \oplus (a \wedge b)) \leq 2kd(a \triangle b, 0) = 2k\tilde{\eta}(a \triangle b) < 2k\delta \leq \varepsilon$, so that $(a, b) \in V_\varepsilon$. Therefore $F_\delta^\triangle \subseteq V_\varepsilon$.

Now let $(a, b) \in V_\delta$. Recall that, by 2.12.1, $(a \triangle b) \ominus ((a \vee b) \ominus a) = a \ominus (a \wedge b)$ and, by 2.12.1, $(a \ominus (a \wedge b)) \wedge (b \ominus (a \wedge b)) = 0$. Hence, applying first the triangle inequality and then 4.3, 4.3, 4.3 and again 4.3, we obtain $\tilde{\eta}(a\triangle b) = d(a\triangle b, 0) \leq d(a\triangle b, (a \vee b) \ominus a) + d((a \vee b) \ominus a, 0) = d(((a \vee b) \ominus a) \oplus (a \ominus (a \wedge b)), (a \vee b) \ominus a) + d((a \vee b) \ominus a, (a \vee a) \ominus a) \leq kd(a \ominus (a \wedge b), 0) + md(a, b) = kd((a \ominus (a \wedge b)) \wedge (a \ominus (a \wedge b)), (b \ominus (a \wedge b)) \wedge (a \ominus (a \wedge b))) + m\, d(a, b) \leq kd(a \ominus (a \wedge b), b \ominus (a \wedge b)) + md(a, b) \leq kmd(a, b) + md(a, b) < (k + 1)m\delta \leq \varepsilon$ so that $(a, b) \in F_\varepsilon^{\triangle}$. We conclude that $V_\delta \subset F_\varepsilon^{\triangle}$. □

Recall that if $\mathbb{G}$ is a topological Abelian group, then a mapping $\mu: L \to \mathbb{G}$ is called a *modular measure* if the following hold, for all $a, b \in L$:

(M1) $\mu(a) + \mu(b) = \mu(a \vee b) + \mu(a \wedge b)$.
(M2) If $a \perp b$, then $\mu(a \oplus b) = \mu(a) + \mu(b)$.

Moreover, the sets $\{(a, b) \in L \times L : \forall r \leq a\triangle b \quad \mu(r) \in W\}$, where W is a neighbourhood of 0 in $\mathbb{G}$, form a base for a D-uniformity $\mathcal{U}$ (see Avallone, 2001, Theorem 3.2). This $\mathcal{U}$ is called the D-uniformity *generated by* μ. Note that, in case μ is positive real-valued (hence in particular a submeasure), $\mathcal{U}$ agrees with the $\mathcal{U}(\mu)$ constructed in Proposition 2.

Theorem 4.4. *Let $\mathcal{U}$ be a D-uniformity on L. Then:*

(a) For every $k > 1$ there is a family $\{\tilde{\eta}_\lambda\}_{\lambda\in\Lambda}$ of k-submeasures with $\mathcal{U} = \sup_{\lambda\in\Lambda}\mathcal{U}(\tilde{\eta}_\lambda)$. Moreover, if $\mathcal{U}$ has a countable base, we can choose $|\Lambda| = 1$.
(b) If $\mathcal{U}$ is generated by a modular measure $\mu: L \to \mathbb{G}$, where $\mathbb{G}$ is a topological Abelian group, then there is a family $\{\tilde{\eta}_\lambda\}_{\lambda\in\Lambda}$ of submeasures with $\mathcal{U} = \sup_{\lambda\in\Lambda}\mathcal{U}(\tilde{\eta}_\lambda)$.
(c) If L is an MV-algebra, there is a family $\{\tilde{\eta}_\lambda\}_{\lambda\in\Lambda}$ of submeasures with $\mathcal{U} = \sup_{\lambda\in\Lambda}\mathcal{U}(\tilde{\eta}_\lambda)$.

Proof:

(a) For every $a, b \in L$, put $f(a, b) = a \wedge b$, $g(a, b) = (a \wedge b^\perp) \oplus b$ and $h(a, b) = (a \vee b) \ominus b$. By (Weber, 1993, Prop. 1.1(b)), $\mathcal{U}$ has base consisting of sets U such that, for every $(a, a') \in U$ and every $b \in L$, $(f(a, b), f(a', b)) = (f(b, a), f(b, a')) \in U$. Since g and h are $\mathcal{U}$-uniformly continuous, from (Weber, 1993, Prop. 1.2) it follows that $\mathcal{U}$ is generated by a family $\{d_\lambda\}_{\lambda\in\Lambda}$ of pseudometrics (a single pseudometric if Λ is countable) such that, for every $\lambda \in \Lambda$ and all $a, a', b, b' \in L$:

$$d_\lambda(f(a, b),\ f(a', b')) \leq d_\lambda(a, a') + d_\lambda(b, b'),$$

$$d_\lambda(g(a, b),\ g(a', b')) \leq k(d_\lambda(a, a') + d_\lambda(b, b')),$$

$$d_\lambda(h(a, b),\ h(a', b')) \leq k(d_\lambda(a, a') + d_\lambda(b, b')).$$

Clearly each d_λ satisfies 4.3 and 4.3, as well as 4.3 with $m = k$, hence also 4.3. Therefore, applying Proposition 4, the conclusion follows.

(b) Let $\{p_\lambda\}_{\lambda\in\Lambda}$ be a family of group seminorms generating the topology of $\mathbb{G}$. By (Fleischer and Traynor, 1982, Theorem 3), $\mathcal{U}$ is generated by the family of pseudometrics $\{d_\lambda\}_{\lambda\in\Lambda}$ where, for every $\lambda \in \Lambda$,

$$d_\lambda(a, b) = \sup\{\, p_\lambda(\mu(r) - \mu(s)) : r, s \in [a \wedge b, a \vee b]\,\}.$$

Moreover d_λ satisifies 4.3 and the following:

$$\forall a, b, c \in L \quad d_\lambda(a \vee c, b \vee c) \leq d_\lambda(a, b). \tag{3}$$

We can complete the proof, applying Proposition 4.3, once we have shown that each d_λ satisfies both 4.3 and 4.3 with $m = k = 1$, hence also 4.3.

Fix $\lambda \in \Lambda$. Given $a, b \in L$, observe first that $d_\lambda(a, b) = \sup\{\, p_\lambda(\mu(r)) : r \leq a\triangle b\,\}$. Now let $c \in L$. By Lemma 3.2 we have $((a \vee c) \ominus c)\triangle((b \vee c) \ominus c) = (a \vee c)\triangle(b \vee c)$. Therefore, by 3, $d_\lambda((a \vee c) \ominus c), ((b \vee c) \ominus c) = d_\lambda(a \vee c, b \vee c) \leq d_\lambda(a, b)$. Finally, if $c \perp a$ and $c \perp b$, then, again by 2, we have $(a \oplus c)\triangle(b \oplus c) = a\triangle b$. Hence $d_\lambda(a \oplus c, b \oplus c) = d_\lambda(a, b)$.

(c) Define f, g and h as in the proof of 4. By (Weber, 1993, Prop. 1.5), since g is associative and distributive with respect to f, the uniformity $\mathcal{U}$ has a base consisting of sets U such that, for every $(a, a') \in U$ and every $b \in L$, $(f(a, b),\ f(a', b)) = (f(b, a),\ f(b, a')) \in U$ and $(g(a, b),\ g(a', b)) = (g(b, a),\ g(b, a')) \in U$. Moreover h is $\mathcal{U}$-uniformly continuous, and therefore from (Weber, 1993, Prop. 1.2) it follows that, for any $m > 1$, $\mathcal{U}$ is generated by a family $\{d_\lambda\}_{\lambda\in\Lambda}$ of pseudometrics (a single pseudometric if Λ is countable) such that, for every $\lambda \in \Lambda$ and all $a, a', b, b' \in L$:

$$d_\lambda(f(a, b),\ f(a', b')) \leq d_\lambda(a, a') + d_\lambda(b, b'),$$

$$d_\lambda(g(a, b),\ g(a', b')) \leq (d_\lambda(a, a') + d_\lambda(b, b')),$$

$$d_\lambda(h(a, b),\ h(a', b')) \leq m(d_\lambda(a, a') + d_\lambda(b, b')).$$

Clearly each d_λ satisfies 4.3, 4.3 with $k = 1$ and 4.3. It remains to show that 4.3 with $k = 1$ is satisified, too. Let $a, c \in L$. By 4.3, we have $d_\lambda((a \vee c) \ominus c, 0) = d_\lambda(a \ominus (a \wedge c), 0) = d_\lambda((a \wedge (a \ominus (a \wedge c)), 0 \wedge (a \ominus (a \wedge c))) \leq d_\lambda(a, 0)$. □

The reader should note that 4.44.4 was already proved in Barbieri and Weber (1998, Theorem 2.5).

REFERENCES

Avallone, A. (2001). Lattice uniformities on orthomodular structures. *Mathematica Slovaca* **51**, 403–419.

Avallone, A., Barbieri, G., and Vitolo, P. (2003). Hahn decomposition of modular measures and applications. *Annal. Soc. Math. Polon.* Series 1: Comment Math. XLIII, 149–168.

Avallone, A. and Basile, A. (2003). On a Marinacci uniqueness theorem for measures. *Journal of Mathematical Analysis and Applications* **286**(2), 378–390.

Avallone, A. and Vitolo, P. (2003). Ideals and congruences of effect algebras. *Order* **20**, 67–77.

Avallone, A. and Weber, H. (1997). Lattice uniformities generated by filters. *Journal of Mathematical Analysis and Applications* **209**(2), 507–528.

Barbieri, G. and Weber, H. (1998). A topological approach to the study of fuzzy measures. In *Functional Analysis and Economic Theory, Samos, 1996*, Springer Verlag, Berlin, pp. 17–46.

Beltrametti, E. G. and Cassinelli, G. (1981). *The Logic of Quantum Mechanics*, Addison-Wesley, Reading, MA.

Bennet, M. K. and Foulis, D. J. (1994). Effect algebras and unsharp quantum logics. *Foundations of Physics* **24**(10), 1331–1352.

Butnariu, D. and Klement, P. (1993). *Triangular Norm-based Measures and Games With Fuzzy Coalitions*. Kluwer, Dordrecht.

Dvurečenskij, A. and Pulmannová, S. (2000). *New Trends in Quantum Structures*. Kluwer, Dordrecht.

Epstein, L. G. and Zhang, J. (2001). Subjective probabilities on subjectively unambiguous events. *Econometrica* **69**(2), 265–306.

Fleischer, I. and Traynor, T. (1982). Group-valued modular functions. *Algebra Universalis* **14**, 287–291.

Graziano, M. G. (2000). Uniformities of Fréchet-Nikodym type on Vitali spaces. *Semigroup Forum* **61**(1), 91–115.

Weber, H. (1991). Uniform lattices. I: A generalization of topological Riesz spaces and topological Boolean rings. *Annali di Matematica Pura e Applicata (4)* **160**, 347–370.

Weber, H. (1993). Metrization of uniform lattices. *Czechoslovak Mathematical Journal* **118**, 271–280.

Weber, H. (1995). Lattice uniformities and modular functions on orthomodular lattices. *Order* **12**, 295–305.